LECTURE NO 23

Electrostatics

METHOD OF IMAGES

The method of images, introduced by Lord Kelvin in 1848, is commonly used to determine V, $\mathbf{E}$, $\mathbf{D}$, and ρ_S due to charges in the presence of conductors. By this method, we avoid solving Poisson's or Laplace's equation but rather utilize the fact that a conducting surface is an equipotential. Although the method does not apply to all electrostatic problems, it can reduce a formidable problem to a simple one.

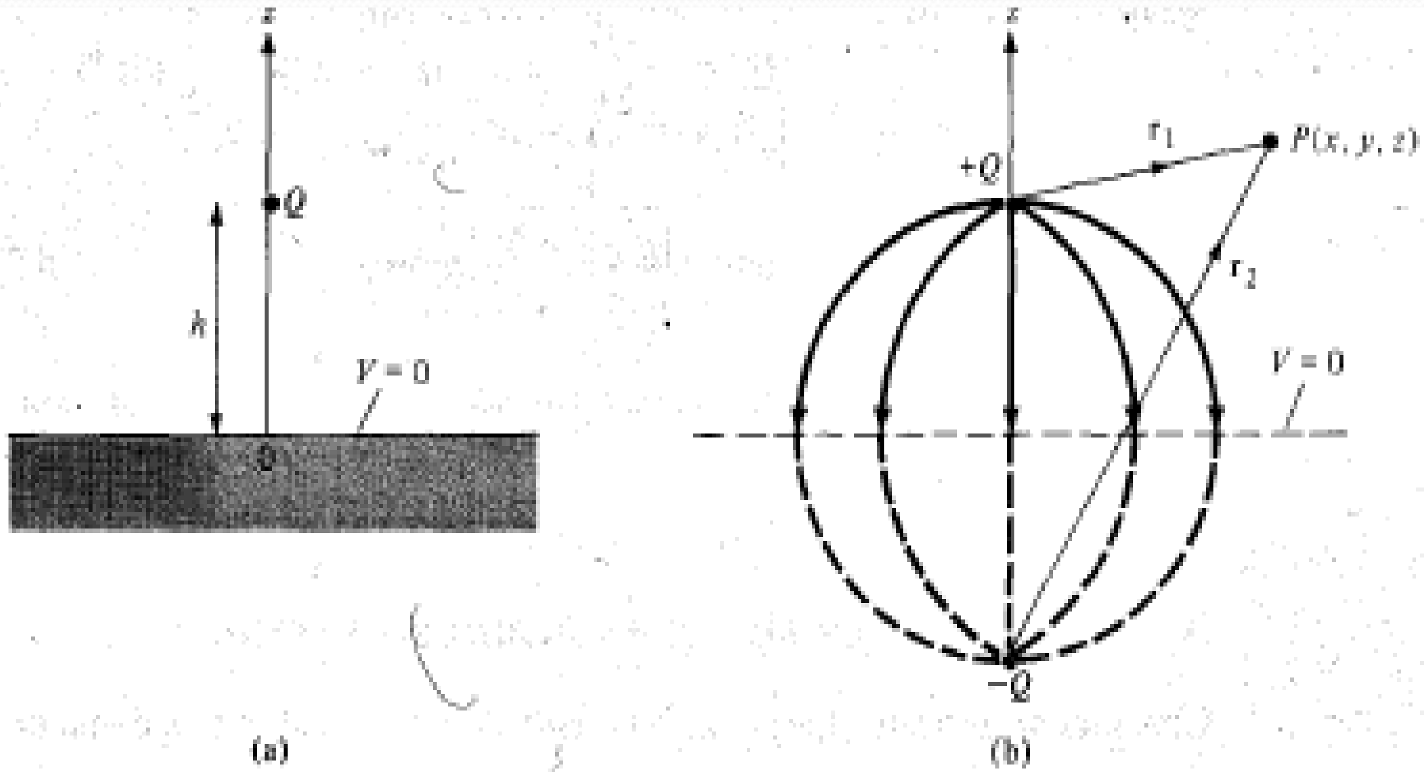

Figure 6.22 (a) Point charge and grounded conducting plane, (b) image configuration and field lines.

A. A Point Charge Above a Grounded Conducting Plane

Consider a point charge Q placed at a distance h from a perfect conducting plane of infinite extent as in Figure 6.22(a). The image configuration is in Figure 6.22(b). The electric field at point $P(x, y, z)$ is given by

$$\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- \tag{6.40}$$

$$= \frac{Q\,\mathbf{r}_1}{4\pi\varepsilon_o r_1^3} + \frac{-Q\,\mathbf{r}_2}{4\pi\varepsilon_o r_2^3} \tag{6.41}$$

The distance vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are given by

$$\mathbf{r}_1 = (x, y, z) - (0, 0, h) = (x, y, z - h) \tag{6.42}$$

$$\mathbf{r}_2 = (x, y, z) - (0, 0, -h) = (x, y, z + h) \tag{6.43}$$

so eq. (6.41) becomes

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_o}\left[\frac{x\mathbf{a}_x + y\mathbf{a}_y + (z-h)\mathbf{a}_z}{[x^2 + y^2 + (z-h)^2]^{3/2}} - \frac{x\mathbf{a}_x + y\mathbf{a}_y + (z+h)\mathbf{a}_z}{[x^2 + y^2 + (z+h)^2]^{3/2}}\right] \tag{6.44}$$

It should be noted that when $z = 0$, $\mathbf{E}$ has only the z-component, confirming that $\mathbf{E}$ is normal to the conducting surface.

The potential at P is easily obtained from eq. (6.41) or (6.44) using $V = -\int \mathbf{E} \cdot d\mathbf{l}$. Thus

$$\begin{aligned} V &= V_+ + V_- \\ &= \frac{Q}{4\pi\varepsilon_o r_1} + \frac{-Q}{4\pi\varepsilon_o r_2} \\ V &= \frac{Q}{4\pi\varepsilon_o}\left\{\frac{1}{[x^2 + y^2 + (z - h)^2]^{1/2}} - \frac{1}{[x^2 + y^2 + (z + h)^2]^{1/2}}\right\} \end{aligned} \tag{6.45}$$

for $z \geq 0$ and $V = 0$ for $z \leq 0$. Note that $V(z = 0) = 0$.

The surface charge density of the induced charge can also be obtained from eq. (6.44) as

$$\begin{aligned} \rho_S = D_n &= \varepsilon_o E_n \Big|_{z=0} \\ &= \frac{-Qh}{2\pi[x^2 + y^2 + h^2]^{3/2}} \end{aligned} \tag{6.46}$$

The total induced charge on the conducting plane is

$$Q_i = \int \rho_S \, dS = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{-Qh \, dx \, dy}{2\pi[x^2 + y^2 + h^2]^{3/2}} \tag{6.47}$$

By changing variables, $\rho^2 = x^2 + y^2$, $dx\,dy = \rho\,d\rho\,d\phi$.

$$Q_i = -\frac{Qh}{2\pi} \int_0^{2\pi} \int_0^{\infty} \frac{\rho \, d\rho \, d\phi}{[\rho^2 + h^2]^{3/2}} \tag{6.48}$$

Integrating over ϕ gives 2π, and letting $\rho\,d\rho = \frac{1}{2} d(\rho^2)$, we obtain

$$\begin{aligned} Q_i &= -\frac{Qh}{2\pi} 2\pi \int_0^{\infty} [\rho^2 + h^2]^{-3/2} \frac{1}{2} d(\rho^2) \\ &= \frac{Qh}{[\rho^2 + h^2]^{1/2}} \Big|_0^{\infty} \\ &= -Q \end{aligned} \tag{6.49}$$

as expected, because all flux lines terminating on the conductor would have terminated on the image charge if the conductor were absent.